

# On a Class of Meromorphic Functions and Their Derivatives

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Abstract: Let  $F$  be a meromorphic function with the order of  $F'/F$  less than  $1/2$ , and the quotient  $(p(z)F''(z) - Q(z)F'(z))/F$  be a rational function for some polynomials  $p(z)$  and  $Q(z)$ , where  $p(z)$  and  $Q(z)$  are not both identically zero. Then

$$F(z) = \frac{P_1(z)}{P_2(z)} e^{P_3(z)},$$

where  $P_i(z)$ ,  $i = 1, 2, 3$ , are polynomials.

It was shown by Csillag [1] that if  $f$  is an entire function and if  $f(z) f^{(p)}(z) f^{(m)}(z) \neq 0$  where  $m > p > 0$ , then  $f = e^{az+b}$ . Hayman [2] proved that every entire function  $f(z)$  for which  $ff'' \neq 0$  has that form. However, the above results are contained in the following stronger

**THEOREM 1.** (Clunie [3]). Suppose that  $f$  is meromorphic and has only a finite number of poles in the plane, and that  $f(z)$  and  $f^{(n)}(z)$  have only a finite number of zeros for some  $n \geq 2$ . Then

$$f(z) = \frac{P_1(z)}{P_2(z)} e^{P_3(z)},$$

where  $P_1, P_2, P_3$  are polynomials. Further, if  $f(z)$  and  $f^{(n)}(z)$  have no zeros then  $f(z) = e^{az+b}$  or  $f(z) = (az + b)^{-n}$ .

In the case  $n = 2$  by means of a result of Wittich [4], it is possible to remove the assumption on the finiteness of the zeros and poles in Theorem 1 and to obtain the same conclusion for a certain class of meromorphic functions. We begin with the following Lemma:

**LEMMA 1.** (Wittich [4]). The rational functions are the only meromorphic functions of order  $< 1/2$ , which satisfy the differential equation

$$(1) \quad W' = R(z, W), \quad (1)$$

where  $R(z, W)$  is a rational function in  $z$  and  $W$ .

**THEOREM 2.** Let  $F$  be a meromorphic function with the order of  $F'/F < 1/2$ . Assume that the quotient  $(p(z)F''(z) - Q(z)F'(z))/F$  is a rational function for some polynomials  $p(z)$  and  $Q(z)$ , where  $p(z)$  and  $Q(z)$  are not both identically zero. Then

$$F(z) = \frac{P_1(z)}{P_2(z)} e^{P_3(z)},$$

where  $P_1(z)$ ,  $P_2(z)$ ,  $P_3(z)$  are polynomials.

*Proof.* Consider the function  $f = F'/F$ . Then

$$\frac{F''}{F} = \left(\frac{F'}{F}\right)^2 + \left(\frac{F'}{F}\right)' = f^2 + f'. \quad (2)$$

By the assumptions, we obtain the equation

$$Q_0(z) = (p(z)F'' - Q(z)F')/F = Q_1(z)f^2 + Q_2(z)f' + Q_3(z)f, \quad (3)$$

where  $Q_1$ ,  $Q_2$ ,  $Q_3$ ,  $Q_0$  are polynomials.

According to Lemma 1, we can conclude

$$f(z) = R(z), \quad (4)$$

where  $R(z)$  is a rational function.

From this it is easy to show that

$$F(z) = \frac{P_1(z)}{P_2(z)} e^{P_3(z)}. \quad (5)$$

**COROLLARY 1.** *If the order of  $F$  is less than  $1/2$ , then for any pair of polynomials  $p(z)$  and  $Q(z)$ , the quotient  $(p(z)F'' - Q(z)F')/F$  cannot be a rational function unless  $F$  itself is a rational function (or both  $p(z)$  and  $Q(z)$  are identically zero).*

*Remark.* The function  $F = \cos \sqrt{z}$  reveals that the restriction of the order in our corollary is the best possible.

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